6.2 - Solutions About Ordinary Points

Consider the differential equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$, which in standard form is $y'' + P(x)y' + Q(x)y = 0$.

Definition: A point $x = x_0$ is an **ordinary point** if P and Q are analytic at x_0 (meaning power series representations centered at $x = x_0$) exist. A point that is not ordinary is **singular**.

Theorem: If $x = x_0$ is an ordinary point of $a_2(x)y''+a_1(x)y'+a_0(x)y = 0$, then we can always find two linearly independent solutions of the form $y = \sum_{n=0}^{\infty} c_n(x-x_0)^n$. For us, $x_0 = 0$

Example: Find two power series solutions of the given differential equation about the ordinary point $x = 0$.

$y'' + 2xy' + 2y = 0$

$$y = \sum_{n=0}^{\infty} c_n x^n, \quad y' = \sum_{n=1}^{\infty} n c_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + 2x \sum_{n=1}^{\infty} n c_n x^{n-1} + 2\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=1}^{\infty} 2n c_n x^n + \sum_{n=0}^{\infty} 2c_n x^n = 0$$

• Exponents x^k

$$\sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k + \sum_{k=1}^{\infty} 2k c_k x^k + \sum_{k=0}^{\infty} 2c_k x^k = 0$$

• Indices

$$\underbrace{2c_2}_{k=0} + \underbrace{2c_0}_{k=0} + \sum_{k=1}^{\infty} \left[(k+2)(k+1)c_{k+2} + (2k+2)c_k\right] x^k = 0$$

$2c_2 + 2c_0 = 0,$ $\quad (k+2)(\cancel{k+1})c_{k+2} + 2(\cancel{k+1})c_k = 0, \quad k=1,2,3,\ldots$

$c_2 = -c_0$ $\quad \boxed{c_{k+2} = -\frac{2}{k+2}c_k, \quad k=1,2,3,\ldots}$

recurrence relation

$k=1 \quad c_3 = -\frac{2}{3}c_1$

$k=2 \quad c_4 = -\frac{1}{2}c_2 = \frac{1}{2}c_0$

$k=3 \quad c_5 = -\frac{2}{5}c_3 = \frac{4}{15}c_1$

$k=4 \quad c_6 = -\frac{1}{3}c_4 = -\frac{1}{6}c_0$

$k=5 \quad c_7 = -\frac{2}{7}c_5 = -\frac{8}{105}c_1$

Recall that $y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \ldots$

Here, $y = c_0 + c_1x - c_0x^2 - \frac{2}{3}c_1x^3 + \frac{1}{2}c_0x^4 + \frac{4}{15}c_1x^5 + \cdots$

$y = c_0 - c_0x^2 + \frac{1}{2}c_0x^4 + \cdots + c_1x - \frac{2}{3}c_1x^3 + \frac{4}{15}c_1x^5 + \cdots$

$y = c_0\left(1 - x^2 + \frac{1}{2}x^4 + \cdots\right) + c_1\left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5 + \cdots\right)$

$$\boxed{y_1 = 1 - x^2 + \frac{1}{2}x^4 + \cdots, \quad y_2 = x - \frac{2}{3}x^3 + \frac{4}{15}x^5 + \cdots}$$

because $y = c_1y_1 + c_2y_2$

Example: Find two power series solutions of the given differential equation about the ordinary point $x = 0$.

Find the recurrence relation and then stop.

$(x+2)y'' + xy' - y = 0$

$$y=\sum_{n=0}^{\infty} c_n x^n, \quad y'=\sum_{n=1}^{\infty} n c_n x^{n-1}, \quad y''=\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$$

$$(x+2)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=1}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1} + \sum_{n=2}^{\infty} 2n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{k=1}^{\infty} k(k+1)c_{k+1}x^k + \sum_{k=0}^{\infty} 2(k+2)(k+1)c_{k+2}x^k + \sum_{k=1}^{\infty} k c_k x^k - \sum_{k=0}^{\infty} c_k x^k = 0$$

$$4c_2 - c_0 = 0, \quad k(k+1)c_{k+1} + 2(k+2)(k+1)c_{k+2} + (k-1)c_k = 0$$

$$c_2 = \frac{1}{4}c_0 \qquad \boxed{c_{k+2} = -\frac{k}{2(k+2)}c_{k+1} - \frac{k-1}{2(k+2)(k+1)}c_k, \quad k=1,2,3,\ldots}$$

$k=1 \quad c_3 = -\frac{1}{6}c_2 = -\frac{1}{24}c_0$

$k=2 \quad c_4 = -\frac{1}{4}c_3 - \frac{1}{24}c_2$

$\qquad = \frac{1}{96}c_0 + \frac{1}{96}c_0 = 0$

$k=3 \quad c_5 = -\frac{3}{10}\cancel{c_4}^{0} - \frac{1}{20}c_3 = \frac{1}{480}c_0$

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

so $\quad y = c_0 + c_1 x + \frac{1}{4}c_0 x^2 - \frac{1}{24}c_0 x^3 + \frac{1}{480}c_0 x^5 + \cdots$

$$\boxed{y_1 = 1 + \frac{1}{4}x^2 - \frac{1}{24}x^3 + \cdots \qquad y_2 = x}$$

Example: Find two power series solutions of the given differential equation about the ordinary point $x = 0$.

recurrence relation

$y'' - (x+1)y' - y = 0$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} k c_k x^k - \sum_{k=0}^{\infty} (k+1)c_{k+1}x^k - \sum_{k=0}^{\infty} c_k x^k = 0$$

$k=0$ terms:

$$2c_2 - c_1 - c_0 = 0$$

$$c_2 = \frac{1}{2}c_1 + \frac{1}{2}c_0$$

$k \geq 1$ terms:

$$(k+2)\cancel{(k+1)}c_{k+2} - \cancel{(k+1)}c_k - \cancel{(k+1)}c_{k+1} = 0$$

$$c_{k+2} = \frac{1}{k+2}c_{k+1} + \frac{1}{k+2}c_k, \; k = 1, 2, 3 \ldots$$

$\cancel{k=1:\ c_3 = \frac{1}{3}c_2 + \frac{1}{3}c_1} = \frac{1}{6}c_1 + \frac{1}{6}c_0 + \frac{1}{3}c_1 = \frac{1}{6}c_0 + \frac{1}{2}c_1$

Let $c_1 = 0$

Then $c_2 = \frac{1}{2}c_0$

$k=1: \quad c_3 = \frac{1}{3}c_2 + \frac{1}{3}c_1^{\nearrow 0}$

$= \frac{1}{6}c_0$

$k=2: \quad c_4 = \frac{1}{4}c_3 + \frac{1}{4}c_2$

$= \frac{1}{24}c_0 + \frac{1}{8}c_0$

$= \frac{1}{6}c_0$

Let $c_0 = 0$

Then $c_2 = \frac{1}{2}c_1$

$k=1: \quad c_3 = \frac{1}{3}c_2 + \frac{1}{3}c_1$

$= \frac{1}{6}c_1 + \frac{1}{3}c_1 = \frac{1}{2}c_1$

$k=2: \quad c_4 = \frac{1}{4}c_3 + \frac{1}{4}c_2$

$= \frac{1}{8}c_1 + \frac{1}{8}c_1 = \frac{1}{4}c_1$

$y = c_0 + c_1 x + c_2 x^2 + \cdots$

$$y = c_0\left(1 + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots\right) + c_1\left(x + \tfrac{1}{2}x^2 + \tfrac{1}{2}x^3 + \cdots\right)$$

$$\boxed{y_1 = 1 + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots, \quad y_2 = x + \tfrac{1}{2}x^2 + \tfrac{1}{2}x^3 + \cdots}$$

Suppose we had initial conditions

$y(0) = a, \quad y'(0) = b$

Then $c_0 = a \qquad c_1 = b$

Use an appropriate Maclaurin series to find two power series solutions of the given differential equation about the ordinary point $x = 0$.

$y'' + e^x y' - y = 0$

Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots\right)$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \left(1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\cdots\right)\sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \cdots$$
$$+ \left(1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\cdots\right)\left(c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots\right)$$
$$- c_0 - c_1 x - c_2 x^2 - c_3 x^3 + \cdots = 0$$

$$2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3$$
$$c_1 x + 2c_2 x^2 + 3c_3 x^3 + 4c_4 x^4 + \frac{1}{2}c_1 x^2 + c_2 x^3 + \frac{3}{2}c_3 x^4$$
$$\frac{1}{6}c_1 x^3 + \frac{1}{3}c_2 x^4 - c_0 - c_1 x - c_2 x^2 - c_3 x^4 + \cdots = 0$$

Constant: $2c_2 + c_1 - c_0 = 0 \Rightarrow c_2 = -\frac{1}{2}c_1 + \frac{1}{2}c_0$

x terms: $6c_3 + 2c_2 + \cancel{c_1} \cancel{- c_1} = 0 \Rightarrow c_3 = -\frac{1}{3}c_2 = \frac{1}{6}c_1 - \frac{1}{6}c_0$

x^2 terms: $12c_4 + 3c_3 + 2c_2 + \frac{1}{2}c_1 - c_2 = 0$

$$c_4 = -\frac{1}{4}c_3 - \frac{1}{12}c_2 - \frac{1}{24}c_1$$
$$= -\frac{1}{4}\left(\frac{1}{6}c_1 - \frac{1}{6}c_0\right) - \frac{1}{12}\left(-\frac{1}{2}c_1 + \frac{1}{2}c_0\right) - \frac{1}{24}c_1$$

and so on

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

$$y = c_0 + c_1 x + \left(-\tfrac{1}{2}c_1 + \tfrac{1}{2}c_0\right)x^2 + \left(\tfrac{1}{6}c_1 - \tfrac{1}{6}c_0\right)x^3 + \cdots$$

$$y = c_0 - \tfrac{1}{2}c_0 x^2 - \tfrac{1}{6}c_0 x^3 + \cdots + c_1 x - \tfrac{1}{2}c_1 x^2 + \tfrac{1}{6}c_1 x^3 + \cdots$$

$$\boxed{y_1 = 1 - \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 + \cdots \quad , \quad y_2 = x - \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots}$$